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Construction of universal branched coverings

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Abstract

A construction for the classifying spaces for branched coverings with branch set a codimension 2 submanifold is given by Brand (1978, 1980). Using this result as a first step we inductively construct universal branched coverings with branch set a stratified set. We also give some of the lower homotopy groups of the classifying spaces which correspond to branched coverings of spheres.

Keywords: Manifolds; Branched coverings; Classification spaces; Normal bundles; Universal bundles; Tubular neighbourhoods

AMS classification: 57M12; 57N80; 55Q52; 55R10

1. Introduction

Since 1920, when Alexander (see [1]) showed that any n-manifold can be considered as a branched covering over the n-sphere, the study of branched coverings has been considered as a fundamental piece in the knowledge of n-manifolds.

Let N^n be a smooth manifold, and let l be a natural number, we say that $K \subset N$ is a *stratified set in* N *of type* l if there is a sequence $\emptyset = K_{l+1} \subset K_l \subset K_{l-1} \subset K_{l-2} \subset \cdots \subset K_3 \subset K_2 = K$ of closed sets in N, such that $(K_j - K_{j+1})$ is a smooth manifold of codimension j and $\overline{K_j - K_{j+1}} = K_j$ for every j, $j = 2, \ldots, l$. The manifolds $(K_j - K_{j+1})$ are called the *strata* of K.

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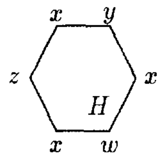

Fig. 1.

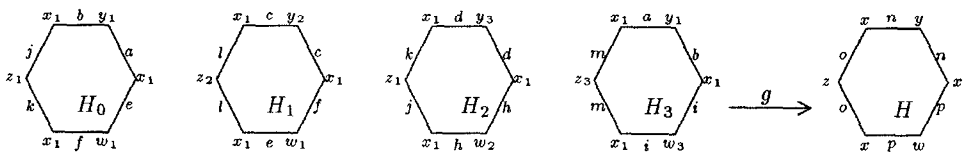

Fig. 2.

Example 1.1. (a) Let K be the surface of a 3-simplex in $\mathbb{R}^4$ with vertex set $\{x_1, x_2, x_3, x_4\}$, considered as a subset of $\mathbb{R}^4$. Then K is a stratified set of type 4 in $\mathbb{R}^4$.

(b) Let K be any knot in S^3, then K is a stratified set of type 2 in S^3.

(c) A graph in $\mathbb{R}^3$ is a stratified set of type 3 in $\mathbb{R}^3$.

Observe that if a set K is already a stratified set of type l, then it is also "trivially" a stratified set of type s, for any s greater than l.

Definition 1.2. Let $\widetilde{N}$, N be smooth compact manifolds, possibly with boundary, let k, l be natural numbers, and let $f : \widetilde{N} \to N$ be a smooth function. We say that f is a *k-fold branched covering of type l over N* if there is a stratified set K of type l in N, called the *branch set*, such that:

(i) $f|_{f^{-1}(N-K)}$ is a k-fold covering.

(ii) $f|_{f^{-1}(K_l)}, f|_{f^{-1}(K_{l-1}-K_l)}, \dots, f|_{f^{-1}(K_2-K_3)}$ are k_s-coverings over their respective components, where k_s is less than k.

(iii) f is transverse to each stratum of K.

(iv) The set $f^{-1}(K)$ is also a stratified set of type l whose strata are the submanifolds $f^{-1}(K_j - K_{j+1})$, $j = 2, \dots, l$.

Notice that the second condition implies that for every $x \in K$, the cardinal of $f^{-1}(x)$ is less than k.

Recall that given M, N smooth manifolds, Z a submanifold of N and $f : M \to N$ a smooth function we say that f is *transverse to* Z if every $x \in f^{-1}(Z)$ satisfies the equation $\mathrm{d}f_x(T_x(M)) + T_{f(x)}(Z) = T_{f(x)}(N)$, where $T_x(M)$ is the tangent space of M at x (see [6]). In this case, it is known that if the codimension of Z is r, then $f^{-1}(Z)$ is a submanifold of M of codimension r. In this sense, condition (iii) will guarantee the fact that $f^{-1}(K_j - K_{j+1})$ is a submanifold in $\widetilde{N}$ of codimension j.

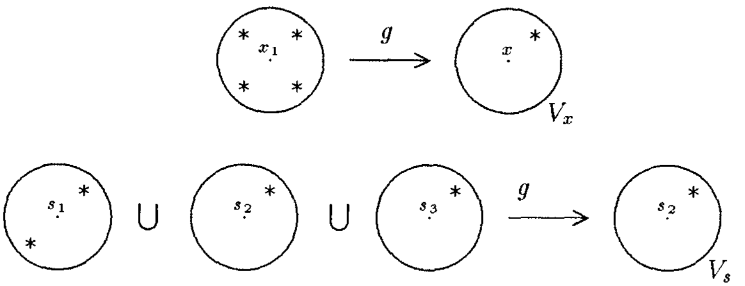

Fig. 3. $s \in \{y, w, z\}$.

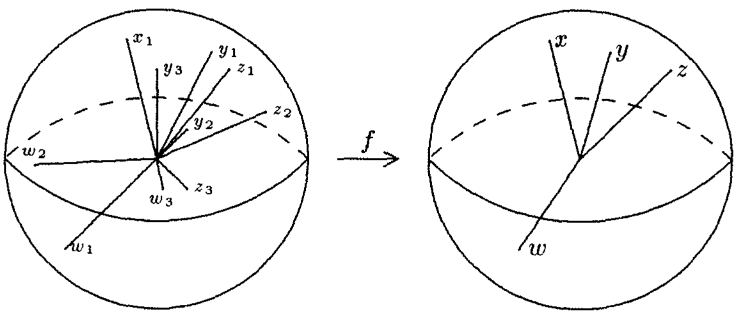

Fig. 4.

Example 1.3. (a) The branched coverings $f : M^n \to S^n$, as Alexander describes in [1], produces k-fold branched coverings of type l over the sphere S^n, where k and l depend on the triangulation used for M^n.

(b) Let H be the hexagon of Fig. 1. Let H_0, H_1, H_2, H_3 be four copies of H. Let $g_i : H_i \to H$ be the natural functions, $i = 0, \ldots, 3$. Let us define $\widetilde{N} = H_0 \cup H_1 \cup H_2 \cup H_3/\sim$, where $\sim$ means that we identify the sides of the first four hexagons as the letters indicate in Fig. 2. Since $\widetilde{N}$ is clearly orientable, calculating its Euler characteristic, we see that $\widetilde{N} \cong S^2$. Also, if $N = H/\sim$ then $N \cong S^2$, where $\sim$ is showed on the last hexagon of Fig. 2. Now, if we define g as the function $(g_0 \cup g_1 \cup g_2 \cup g_3/\sim) : \widetilde{N} \to N$ then g is a 4-fold branched covering of type 2 over S^2, where $K = \{x, y, z, w\} = K_2$ and K_3 is the empty set. Notice that $g^{-1}(x) = \{x_1\}$, $g^{-1}(y) = \{y_1, y_2, y_3\}$, $g^{-1}(z) = \{z_1, z_2, z_3\}$ and $g^{-1}(w) = \{w_1, w_2, w_3\}$. Fig. 3 illustrates the preimages of small neighbourhoods V_x, V_y, V_z, V_w of x, y, z, w, respectively.

(c) Let $g : S^2 \to S^2$ be the function defined above. Let D^3 be a 3-disk (i.e., a solid ball of radius 1). Let us define $f(z) = |z|^3 g(z/|z|)$, $\forall z \in D^3$. Clearly, $f : D^3 \to D^3$ is an extension of g and, moreover, f is a 4-fold branched covering of type 3 over D^3, where the straight lines in the second ball in Fig. 4 represent the branched set $K = K_2$. Here K_3 reduces to the center of D^3. Compare with Example 4.1.

(d) Let $\widetilde{D}_0, \widetilde{D}_1, D_0, D_1$ be four copies of D^3, and let $f_0 : \widetilde{D}_0 \to D_0$, $f_1 : \widetilde{D}_1 \to D_1$ be two copies of the function $f : D^3 \to D^3$ defined in (c). If $\widetilde{N} = \widetilde{D}_0 \cup_{\mathrm{Id}} \widetilde{D}_1$ and $N = D_0 \cup_{\mathrm{Id}} D_1$ (where Id is the identity function) then $\widetilde{N} \cong N \cong S^3$. Now, if $f_0 \cup_{\mathrm{Id}} f_1 : \widetilde{N} \to N$ is the double mapping cylinder then $f_0 \cup_{\mathrm{Id}} f_1$ is a 4-fold branched covering of type 3 over S^3, where the stratum at the deepest level 3 is a set of two points.

Now, extending $f_0 \cup_{\mathrm{Id}} f_1$ radially to the 4-disk D^4 (as we did in (c)), we obtain a 4-fold branched covering of type 4 over D^4, where K_4 is the center of the disk D^4, and if, for this new function, we repeat the process described at the beginning of this item, we will obtain a 4-fold branched covering of type 4 over S^4. Of course, inductively, we could continue this kind of constructions for any l.

(e) Let S_1, S_2, S_3 be three copies of the circle S^1. Let $F_i : S_i \to S^1$ an i-fold covering for $i = 1, 2, 3$. Let $F : (S_1 \cup S_2 \cup S_3) \to S^1$ such that $F|_{S_i} = F_i$. Then F is a 6-fold covering. Consider $g \times F : S^2 \times (S_1 \cup S_2 \cup S_3) \to S^2 \times S_1$, where g is the function constructed in (b). So $g \times F$ is a 24-fold branched covering of type 2 over $S^2 \times S^1$, where the branch set consists of four disjoint circles $x \times S^1, y \times S^1, z \times S^1$, and $w \times S^1$. Notice that $(g \times F)^{-1}(x \times S^1) = x_1 \times (S_1 \cup S_2 \cup S_3)$, and $(g \times F)^{-1}(y \times S^1) = y_1 \times (S_1 \cup S_2 \cup S_3) \cup y_2 \times (S_1 \cup S_2 \cup S_3) \cup y_3 \times (S_1 \cup S_2 \cup S_3)$ the preimages of $z \times S^1$ and $w \times S^1$ are similar to the preimage $(g \times F)^{-1}(y \times S^1)$.

Definition 1.4. Let $f_1 : \widetilde{M}_1 \to M$ and $f_2 : \widetilde{M}_2 \to M$ be k-fold branched coverings of type l over a manifold M. We say that f_1 and f_2 are *equivalent up to homeomorphism* if there is a homeomorphism $h : \widetilde{M}_1 \to \widetilde{M}_2$ such that $f_2 \circ h = f_1$. Now, we say that f_1 and f_2 are *concordant* (of type l) if there is a k-fold branched covering of type l, $F : \widetilde{W}^{n+1} \to M^n \times I$ (I is the closed interval $[0, 1]$), such that $F|_{F^{-1}(M \times \{0\})}$ is equivalent up to homeomorphism with f_1 and $F|_{F^{-1}(M \times \{1\})}$ is equivalent up to homeomorphism with f_2 (see Example 4.1).

A triple (E, B, γ) is called a *k-fold universal branched covering of type l* if $\gamma : E \to B$ is a k-fold branched covering of type l and for any k-fold branched covering of type l, $f : \widetilde{N} \to N$, there is a continuous function $\boldsymbol{c} : N \to B$ such that the pullback of γ under $\boldsymbol{c}$ ($\boldsymbol{c}^*(\gamma)$) is a k-fold branched covering that is concordant with f. Moreover, if $c_1 : N \to B$, $c_2 : N \to B$ are maps such that the pullbacks $\boldsymbol{c}_1^*\gamma$ and $\boldsymbol{c}_2^*\gamma$ give concordant branched coverings then the maps $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are homotopic. B is called a *classifying space* and $\boldsymbol{c}$ is called a *classifying function*.

It is not difficult to see that there is a bijection between the set of concordance classes of k-fold branched coverings of type l over N which is denoted by $BR_k(N)$ and the set $[N, B]$ of homotopic classes of continuous functions from N to B. In this way, any information about the nth-homotopy group of the classifying space B, $\pi_n(B)$, will give knowledge about branched coverings over the n-sphere and by the mentioned Alexander's result it will help in the comprehension of n-manifolds.

Using Brown's representability theorem, Tejada has proven (see [9,10]) the existence of classifying spaces for branched coverings over CW-complexes. Unfortunately her proof is not constructive. One of the goals of the present paper is to exhibit an explicit construction of classifying spaces for branched coverings over manifolds.

The construction of classifying spaces for branched coverings of type 2 over manifolds was done by Brand in [2,3]. If $k \in \mathbb{N}$, let us denote by $(E(2), BR_k(2), \gamma(2))$ the universal k-fold branched covering of type 2. In Sections 2 and 3 we give an inductive construction for $(E(l), BR_k(l), \gamma(l))$ the universal k-fold branched covering of type l. The method consists in taking a tubular neighbourhood around the submanifold that is placed at the deepest level (l) of the stratified branch set. Outside of this tubular neighbourhood the induction hypothesis guarantees the existence of the universal space. Inside of the tubular neighbourhood, a product of universal coverings with universal normal disk bundles gives the classifying space. Those two spaces are glued together by taking a double mapping cylinder to get the classifying space for the k-fold branched coverings of type l. Finally, in Section 4 we give some homotopy groups of the classifying spaces.

2. A local condition

Before starting the construction of the universal branched coverings, we need to give some conditions for K_l and $f^{-1}(K_l)$, and a precise local condition for the fiber of the normal bundles of the strata K_l and $f^{-1}(K_l)$ in N and $\widetilde{N}$. Actually, our universal branched covering will be universal for the family of k-branched coverings of type l that verify the conditions (i), (ii), (iii) and (iv) (below) for fixed sets A and $\{B_a\}_{a\in A}$. Intuitively, it is good to think of $(E(l), BR_k(l), \gamma(l))$ as an infinite photograph file, where we can find the photograph of any possible k-fold branched covering of type l verifying the conditions that we will specify below.

Let us consider Example 1.2(b) again. In Fig. 3 we have illustrated the components of the preimages of small neighbourhoods of x, y, z and w. Their characteristics give in a natural way a partition on the branch set, i.e., $\{x\}, \{y, z, w\}$. Also, we see that $g^{-1}(\{y, z, w\})$ is partitioned in a natural way in two sets in the following way

$$\{\{y_1, z_1, w_1\}, \{y_2, z_2, w_2, y_3, z_3, w_3\}\},$$

while $g^{-1}(\{x\}) = \{x_1\}$.

In general, for a fixed k-fold branched covering of type l, $f : \widetilde{N} \to N$ $(l, k \in \mathbb{N})$, we observe the existence of finite sets A, $\{B_a\}_{a\in A}$ and natural partitions $\{K_{l,a}\}_{a\in A}$, $\{\widetilde{K}_{l,b_a}\}_{b_a\in B_a}$ of K_l and $\widetilde{K}_{l,a} = f^{-1}(K_{l,a})$, respectively, such that:

(i) $f(\widetilde{K}_{l,b_a}) = K_{l,a}$ for every $b_a \in B_a$.

(ii) $f|\widetilde{K}_{l,a} : \widetilde{K}_{l,a} \to K_{l,a}$ is an r_a-fold covering, for some r_a less than k.

(iii) Each $f|_{\widetilde{K}_{l,b_a}} : \widetilde{K}_{l,b_a} \to K_{l,a}$ is an r_{b_a}-fold covering, where $\{r_{b_a}\}_{b_a\in B_a}$ is a fixed partition of r_a.

(iv) The closed disk normal bundles of $\widetilde{K}_{l,b_a}$ and $K_{l,a}$ satisfy the *local condition* that we will specify just after the next examples.

Example 2.1. Let us see these sets and numbers in:

(a) Example 1.2(b): $K_2 = \{x, y, z, w\}$, $A = 1, 2$, $K_{2,1} = \{x\}$, $K_{2,2} = \{y, z, w\}$, $B_1 = \{1_1\}$, $\widetilde{K}_{2,1_1} = \{x_1\}$, $B_2 = \{1_2, 2_2\}$, $\widetilde{K}_{2,1_2} = \{y_1, z_1, w_1\}$, $\widetilde{K}_{2,2_2} = \{y_2, z_2, w_2, y_3, z_3, w_3, \}$. $r_1 = 1 = r_{1_1}$, $r_2 = 3$, $r_{1_2} = 1$, $r_{2_1} = 2$.

(b) Example 1.2(e): K_2 is partitioned in $K_{2,1} = x \times S^1$ and $K_{2,2} = (y \times S^1) \cup (z \times S^1) \cup (w \times S^1)$, the set A is $\{1, 2\}$. Now $\widetilde{K}_{2,1_1} = x_1 \times (S_1 \cup S_2 \cup S_3)$ so $B_1 = \{1_1\}$,

$$\widetilde{K}_{2,1_2} = \big(y_1 \times (S_1 \cup S_2 \cup S_3)\big) \cup \big(z_1 \times (S_1 \cup S_2 \cup S_3)\big) \cup \big(w_1 \times (S_1 \cup S_2 \cup S_3)\big),$$

and

$$\begin{aligned}\widetilde{K}_{2,2_2} = {}& \big(y_2 \times (S_1 \cup S_2 \cup S_3)\big) \cup \big(z_2 \times (S_1 \cup S_2 \cup S_3)\big) \cup \big(w_2 \times (S_1 \cup S_2 \cup S_3)\big) \\ & \cup \big(y_3 \times (S_1 \cup S_2 \cup S_3)\big) \cup \big(z_3 \times (S_1 \cup S_2 \cup S_3)\big) \cup \big(w_3 \times (S_1 \cup S_2 \cup S_3)\big),\end{aligned}$$

so $B_2 = \{1_2, 2_2\}$ and $r_1 = 1 = r_{1_1}$, $r_2 = 3$, $r_{1_2} = 1$, $r_{2_2} = 2$.

For each $a \in A$ and $b_a \in B_a$, let

$$\big(E_{\tilde{\eta}_{l,b_a}}, \widetilde{K}_{l,b_a}, p_{\tilde{\eta}_{l,b_a}}\big) = \tilde{\eta}_{l,b_a} \quad \text{and} \quad \big(E_{\eta_{l,a}}, K_{l,a}, p_{\eta_{l,a}}\big) = \eta_{l,a}$$

be the closed disk normal bundles of $\widetilde{K}_{l,b_a}$ and $K_{l,a}$, respectively. Hence, their fiber is the closed disk D^l with radius 1. If $T\widetilde{K}_{l,b_a}, TK_{l,a}$ are the closure of the open tubular neighbourhoods of $\widetilde{K}_{l,b_a}$ and $K_{l,a}$, respectively, we know (see [6, p. 76] and [7, p. 115]) that the total space $E_{\tilde{\eta}_{l,b_a}}$ of $\tilde{\eta}_{l,b_a}$ is diffeomorphic to $T\widetilde{K}_{l,b_a}$. In the same way: $E_{\eta_{l,a}} \cong TK_{l,a}$. Furthermore,

$$T\widetilde{K}_{l,b_a} = \bigcup_{y \in \widetilde{K}_{l,b_a}} D^l_{\varepsilon(y)}(y)$$

where $D^l_{\varepsilon(y)}(y)$ is a closed disk of radius $\varepsilon(y)$ centered at y, where ε is a smooth positive function on $\widetilde{K}_{l,b_a}$. Similarly

$$TK_{l,a} = \bigcup_{x \in K_{l,a}} D^l_{\varepsilon(x)}(x).$$

Now, let us assume the following *local condition*: Let $a \in A$, for any $x \in K_{l,a}$ and any $y \in f^{-1}(x) \subset \bigcup_{b_a \in B_a} \widetilde{K}_{l,b_a}$ we are going to assume that $f(D^l_{\varepsilon(y)}(y)) = D^l_{\varepsilon(x)}(x)$, and that there are coordinate systems $\phi_x : D^l_{\varepsilon(x)}(x) \to D^l$, $\phi_y : D^l_{\varepsilon(y)}(y) \to D^l$ (where D^l is the closed disk of radius 1 centered at the origin in $\mathbb{R}^l$ and $\phi_x(x) = 0$, $\phi_y(y) = 0$) such that the following diagram is commutative:

$$\begin{tikzcd} D^l_{\varepsilon(y)}(y) \arrow[r, "f"] \arrow[d, "\phi_y"'] & D^l_{\varepsilon(x)}(x) \arrow[d, "\phi_x"] \\ D^l \arrow[r, "f_{0,b_a}"'] & D^l \end{tikzcd} \tag{2.1}$$

where $b_a \in B_a$ is such that $y \in \widetilde{K}_{l,b_a}$ and the function $f_{0,b_a} : D^l \to D^l$ is defined in the following way:

Let S^{l-1} be the $(l-1)$-sphere of radius 1. In order to define f_{0,b_a}, we choose $g_{b_a}: S^{l-1} \to S^{l-1}$ a k_{b_a}-fold branched covering of type $l-1$ (where $\{k_{b_a}\}_{b_a \in B_a}$ is a fixed partition of k), and for any $z \in D^l$ let

$$f_{0,b_a}(z) = |z|^p g_{b_a}\left(\frac{z}{|z|}\right), \quad \text{for some } p \geqslant l.$$

We choose $p \geqslant l$ so that f_{0,b_a} will be differentiable (see Example 1.2(c)). In this way, $f_{0,b_a}: D^l \to D^l$ is a k_{b_a}-fold branched covering of type l for which the branch submanifold of codimension l reduces to the center 0 of the disk D^l. Roughly speaking, we want the function f to be "equal to" f_{0,b_a} for each fiber of the normal bundle $\widetilde{\eta}_{l,b_a}$.

Let us go back to Examples 1.2(e) and 2.1(b). Let (x,s) be any point in $K_{2,1} = x \times S^1$, if $(g \times F)(\widetilde{x}, \widetilde{s}) = (x,s)$, where $(\widetilde{x}, \widetilde{s}) \in \widetilde{K}_{2,1_1}$, we see that the restriction of $g \times F$ to the fiber of the normal bundle of, for example, $\widetilde{K}_{2,1_1}$ at $(\tilde{x}, \tilde{s})$ looks like the complex function $z \mapsto z^4$, $|z| \leqslant 1$, i.e., the function $g \times F$ is locally "the same" along $\widetilde{K}_{2,1_1}$.

Since we want to construct the universal branched covering for the family of k-fold branched coverings of type l that verify the local condition, it is clear that $\gamma(l)$ must also satisfy this condition. Next, we construct a function $(B\widetilde{F}_{b_a})_{D^l}$ that will be equal to $\gamma(l)$, when $\gamma(l)$ is restricted to a small ball that is the fiber, at some point, of the normal bundle of $\widetilde{K}_{l,b_a}$.

Let $\widetilde{G}_{b_a}$, and G_a be the groups of coordinate transformations of the normal bundles $\widetilde{\eta}_{l,b_a}, \eta_{l,a}$, respectively. Because of the commutativity of diagram (2.1) it follows that whenever a coordinate change is made in $\widetilde{\eta}_{l,b_a}$ a corresponding coordinate change must be made in $\eta_{l,a}$. It is not difficult to check that it gives a group homomorphism $\mu_{b_a}: \widetilde{G}_{b_a} \to G_a$ such that the diagram:

$$\begin{tikzcd} D^l \arrow[r, "\tilde{g}_{b_a}"] \arrow[d, "f_{0,b_a}"'] & D^l \arrow[d, "f_{0,b_a}"] \\ D^l \arrow[r, "\mu_{b_a}(\tilde{g}_{b_a})"'] & D^l \end{tikzcd} \tag{2.2}$$

commutes, for every $\tilde{g}_{b_a} \in \widetilde{G}_{b_a}$.

Since $f|_{\widetilde{K}_{l,b_a}}: \widetilde{K}_{l,b_a} \to K_{l,a}$ is an r_{b_a}-fold covering then we get a classifying function $r_{b_a}: K_{l,a} \to B_{S_{r_{b_a}}}$ such that $f|_{\widetilde{K}_{l,b_a}}$ is the pullback in the next diagram:

$$\begin{tikzcd} \widetilde{K}_{l,b_a} \arrow[r, "\tilde{r}_{b_a}"] \arrow[d, "f|_{\widetilde{K}_{l,b_a}}"'] & E_{S_{r_{b_a}}} \arrow[d, "\gamma_{S_{r_{b_a}}}"] \\ K_{l,a} \arrow[r, "r_{b_a}"'] & B_{S_{r_{b_a}}} \end{tikzcd} \tag{2.3}$$

where $S_{r_{b_a}}$ is the symmetric group on r_{b_a} elements and $(E_{S_{r_{b_a}}}, B_{S_{r_{b_a}}}, \gamma_{S_{r_{b_a}}})$ is the universal r_{b_a}-fold covering.

Let $(E_{G_a}, B_{G_a}, p_{G_a})$, $(E_{\widetilde{G}_{b_a}}, B_{\widetilde{G}_{b_a}}, p_{\widetilde{G}_{b_a}})$ be the universal G_a, and $\widetilde{G}_{b_a}$-bundles, respectively (see [7, p. 52]). Because of the action of G_a and $\widetilde{G}_{b_a}$ over D^l we can change the fiber of $(E_{G_a}, B_{G_a}, p_{G_a})$, $(E_{\widetilde{G}_{b_a}}, B_{\widetilde{G}_{b_a}}, p_{\widetilde{G}_{b_a}})$ to be the disk D^l. As usual we denote these disk bundles

$$\big((E_{G_a})_{D^l}, B_{G_a}, (p_{G_a})_{D^l}\big), \quad \big((E_{\widetilde{G}_{b_a}})_{D^l}, B_{\widetilde{G}_{b_a}}, (p_{\widetilde{G}_{b_a}})_{D^l}\big).$$

We get similar notation if we consider the sphere bundles, i.e., if we take the fiber as S^{l-1}.

Notice that the restrictions $f|_{\widetilde{K}_{l,b_a}} = F_{b_a}$ and $f|_{E_{\widetilde{\eta}_{l,b_a}}} = \widetilde{F}_{b_a}$ induce canonical functions $BF_{b_a} : B_{\widetilde{G}_{b_a}} \to B_{Ga}$ and $(B\widetilde{F}_{b_a})_{D^l} : (E_{\widetilde{G}_{b_a}})_{D^l} \to (E_{G_a})_{D^l}$ (see [9, p. 50]) such that the following diagram conmutes:

$$\begin{tikzcd}
(E_{\widetilde{G}_{b_a}})_{D^l} \arrow[r, "(B\widetilde{F}_{b_a})_{D^l}"] \arrow[d, "(p_{\widetilde{G}_{b_a}})_{D^l}"'] & (E_{G_a})_{D^l} \arrow[d, "(p_{G_a})_{D^l}"] \\
B_{\widetilde{G}_{b_a}} \arrow[r, "BF_{b_a}"'] & B_{Ga}
\end{tikzcd} \tag{2.4}$$

Moreover, the restriction of $(B\widetilde{F}_{b_a})_{D^l}$ to the fiber D^l is equivalent to $f_{0,b_a} : D^l \to D^l$ up to homeomorphism. Thus, $(B\widetilde{F}_{b_a})_{D^l}$ is a k_{b_a}-fold branched covering of type l, and $(B\widetilde{F}_{b_a})_{S^{l-1}}$ is a k_{b_a}-fold branched covering of type $l-1$. Since $(B\widetilde{F}_{b_a})_{D^l}$ extends to $\gamma(l)$, we already know that the local condition will be verified by $\gamma(l)$.

3. The construction

We start the construction defining some spaces. The considered topologies are always canonical topologies. Let A, $\{B_a\}_{a\in A}$ be fixed sets. Let $a \in A$ and $b_a \in B_a$. Let $X_{r_{b_a}} = \prod_{c\in B_a} Y_{r_c}$ such that $Y_{r_c} = B_{S_{r_c}}$, for all c different from b_a, and $Y_{r_{b_a}} = E_{S_{r_{b_a}}}$.

Next, we define the function $(\Phi_a)_{D^l}$ to be the restriction of $\gamma(l)$ to the tubular neighbourhood of the preimage of the deepest stratum (level l) of its branch set. If $\dot{\bigcup}$ means disjoint union, let us define

$$(\Phi_a)_{D^l} : \dot{\bigcup_{b_a\in B_a}} \Big(X_{r_{b_a}} \times (E_{\widetilde{G}_{b_a}})_{D^l}\Big) \to \Big(\prod_{b_a\in B_a} B_{S_{r_{b_a}}} \times (E_{G_a})_{D^l}\Big)$$

in the following way: if $x \in \dot{\bigcup}_{b_a\in B_a}(X_{r_{b_a}} \times (E_{\widetilde{G}_{b_a}})_{D^l})$ then there is $b_a \in B_a$ such that $x \in (X_{r_{b_a}} \times (E_{\widetilde{G}_{b_a}})_{D^l})$, in other words, $x = ((x_c)_{c\in B_a}, e)$ where $(x_c)_{c\in B_a} \in X_{r_{b_a}}$ and $e \in (E_{\widetilde{G}_{b_a}})_{D^l}$. Let

$$(\Phi_a)_{D^l}(x) = \big((y_c)_{c\in B_a}, (B\widetilde{F}_{b_a})_{D^l}(e)\big)$$

with $y_c = x_c$ if c is different from b_a and $y_{b_a} = \gamma_{S_{r_{b_a}}}(x_{b_a})$. Clearly $(\Phi_a)_{D^l}$ is a well-defined continuous function. Moreover, since $(B\widetilde{F}_{b_a})_{D^l}$ is a k_{b_a}-fold branched covering of type l, where $\{k_{b_a}\}_{b_a\in B_a}$ is a partition of k , then $(\Phi_a)_{D^l}$ is a k-fold branched covering of type l, and $(\Phi_a)_{S^{l-1}}$ is a k-fold branched covering of type $l-1$.

If the existence of $(E(l-1), BR_k(l-1), \gamma(l-1))$, the universal k-fold branched covering of type $l-1$ is assumed, we get a classifying function

$$\mathbf{c}\colon \dot{\bigcup_{a\in A}}\Bigl(\prod_{b_a\in B_a} B_{S_{r_{b_a}}} \times (E_{G_a})_{S^{l-1}}\Bigr) \to BR_k(l-1)$$

such that $\mathbf{c}^*\gamma(l-1) = \dot{\bigcup}_{a\in A}(\Phi_a)_{S^{l-1}}$, i.e., the following diagram commutes:

$$\begin{array}{ccc} \dot{\bigcup}_{a\in A}\dot{\bigcup}_{b_a\in B_a}\bigl(X_{r_{b_a}} \times (E_{\widetilde{G}_{b_a}})_{S^{l-1}}\bigr) & \xrightarrow{\ \bar{\mathbf{c}}\ } & E(l-1) \\ {\scriptstyle \mathbf{c}^*\gamma(l-1)=\dot{\bigcup}_{a\in A}(\Phi_a)_{S^{l-1}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \gamma(l-1)} \\ \dot{\bigcup}_{a\in A}\bigl(\prod_{b_a\in B_a} B_{S_{r_{b_a}}} \times (E_{G_a})_{S^{l-1}}\bigr) & \xrightarrow[\ \mathbf{c}\ ]{} & BR_k(l-1) \end{array} \tag{3.1}$$

Now, we have all the pieces to construct the universal branched covering. On one hand, we understand the tubular neighbourhoods of the deepest stratum of the branch set and its preimage. We also understand $\gamma(l)$ restricted to these neighbourhoods. On the other hand, on the complement of these neighbourhoods, the universal branched covering coincides with $(E(l-1), BR_k(l-1), \gamma(l-1))$. So we only need to put together the pieces. The next theorem shows how to glue them.

Theorem. *The triple $(E(l), BR_k(l), \gamma(l))$ defined by*

$$BR_k(l) = \Bigl(\dot{\bigcup_{a\in A}}\Bigl(\prod_{b_a\in B_a} B_{S_{r_{b_a}}} \times (E_{G_a})_{D^l}\Bigr)\Bigr) \cup_{\mathbf{c}} BR_k(l-1),$$

$$E(l) = \Bigl(\dot{\bigcup_{a\in A}}\ \dot{\bigcup_{b_a\in B_a}} \bigl(X_{r_{b_a}} \times (E_{\widetilde{G}_{b_a}})_{D^l}\bigr)\Bigr) \cup_{\bar{\mathbf{c}}} E(l-1),$$

$$\gamma(l) = \Bigl(\dot{\bigcup_{a\in A}}(\Phi_a)_{D^l}\Bigr) \cup_{\bar{\mathbf{c}}} \gamma(l-1)$$

is a universal k-fold branched covering of type l for functions that satisfy the conditions mentioned before.

Proof. In fact, let M be a manifold, and let $g\colon M \to BR_k(l)$ be a smooth function. Deforming g slightly we can make g transverse to each stratum (see [9, p. 12]) of the branch set of the next restriction function:

$$\begin{aligned} &\gamma(l)\Big|\Bigl(\dot{\bigcup_{a\in A}}\ \dot{\bigcup_{b_a\in B_a}} \bigl(X_{r_{b_a}} \times (E_{\widetilde{G}_{b_a}})_{D^l}\bigr)\Bigr) \\ &\quad = \dot{\bigcup_{a\in A}}(\Phi_a)_{D^l} : \Bigl(\dot{\bigcup_{a\in A}}\ \dot{\bigcup_{b_a\in B_a}} \bigl(X_{r_{b_a}} \times (E_{\widetilde{G}_{b_a}})_{D^l}\bigr)\Bigr) \\ &\quad \to \Bigl(\dot{\bigcup_{a\in A}}\Bigl(\prod_{b_a\in B_a} B_{S_{r_{b_a}}} \times (E_{G_a})_{D^l}\Bigr)\Bigr). \end{aligned}$$

Since each $(\Phi_a)_{D^l}$ is a k-fold branched covering of type l, the disjoint union function $\dot{\bigcup}_{a\in A}(\Phi_a)_{D^l}$ is also a k-fold branched covering of type l. Let us say that R is its branch set. Thus, R is a stratified set of type l in $(\dot\bigcup_{a\in A}\dot\bigcup_{b_a\in B_a}(X_{r_{b_a}}\times(E_{\widetilde{G}_{b_a}})_{D^l})$. Moreover if R_l is the submanifold of codimension l related to the stratified set R, it is clear that $R_l=(\dot\bigcup_{a\in A}(\prod_{b_a\in B_a}B_{S_{r_{b_a}}}\times B_{G_a})$ where B_{G_a} is the space of zero vectors of $(E_{G_a})_{D^l}$.

Pulling back, we see that $g^{-1}(R_l)$ is a submanifold of M of codimension l (call it K_l). Furthermore, $g^{-1}(\dot\bigcup_{a\in A}\dot\bigcup_{b_a\in B_a}(X_{r_{b_a}}\times(E_{\widetilde{G}_{b_a}})_{D^l})$ gives us a tubular neighbourhood of K_l, let us call it TK_l. Therefore, the restriction function

$$h_{TK_l}=\left(g^*\left(\dot\bigcup_{a\in A}(\Phi_a)_{D^l}\right)\right)\Big|_{\left(g^*\left(\dot\bigcup_{a\in A}(\Phi_a)_{D^l}\right)\right)^{-1}(TK_l)}$$

is a k-fold branched covering over TK_l of type l, where the function $g^*(\dot\bigcup_{a\in A}(\Phi_a)_{D^l})$ is the pullback of $(\dot\bigcup_{a\in A}(\Phi_a)_{D^l})$ under g.

On the other hand, since

$$g^{-1}(BR_k(l-1))=\left(\overline{M-g^{-1}\left(\dot\bigcup_{a\in A}\dot\bigcup_{b_a\in B_a}(X_{r_{b_a}}\times(E_{\widetilde{G}_{b_a}})_{D^l})\right)}\right)=(\overline{M-TK_l})$$

we obtain a k-fold branched covering of type $l-1$ over $(\overline{M-TK_l})$ (see [9, p. 15]), let us call it: $h_{(\overline{M-TK_l})}$. The definition of $BR_k(l)$ implies that h_{TK_l} coincides with $h_{(\overline{M-TK_l})}$ on the inverse image $g^{-1}(\dot\bigcup_{a\in A}\dot\bigcup_{b_a\in B_a}(X_{r_{b_a}}\times(E_{\widetilde{G}_{b_a}})_{S^{l-1}})$. Hence, the pullback $g^*\gamma(l)$ in the following commutative diagram:

$$\begin{tikzcd}
\widetilde{M} \arrow[r] \arrow[d, "g^*\gamma(l)"'] & E(l) \arrow[d, "\gamma(l)"] \\
M \arrow[r, "g"'] & B(l)
\end{tikzcd} \tag{3.2}$$

is a k-fold branched covering of type l over M.

Now, let $f:\widetilde{M}\to M$ be a k-fold branched covering of type l that satisfies the *local condition* mentioned before. We claim that there exists $\boldsymbol{c}':M\to BR_k(l)$ that classifies f. In fact, since $f|_{(\overline{\widetilde{M}-T\widetilde{K}_l})}$ is a k-fold branched covering of type $l-1$ there is $\boldsymbol{c}_0:(\overline{M-TK_l})\to BR_k(l-1)$ that classifies $f|_{(\overline{\widetilde{M}-T\widetilde{K}_l})}$. Let us define $\boldsymbol{c}':M\to BR_k(l)$ in the following way: $\boldsymbol{c}'(z)=\boldsymbol{c}_0(z)$ if $z\in(\overline{M-TK_l})$. Now, if $z\in TK_l=(\dot\bigcup_{a\in A}TK_{l,a})$ then there is $a\in A$ such that $z\in TK_{l,a}=\bigcup_{x\in K_{l,a}}D^l_{\varepsilon(x)}(x)\cong E_{\eta_{l,a}}$. Let

$$\boldsymbol{c}'(z)=\left(\boldsymbol{r}_a(p_{\eta_{l,a}}(z)),\bar{h}(z)\right)\in\left(\dot\bigcup_{a\in A}\left(\prod_{b_a\in B_a}B_{S_{r_{b_a}}}\times(E_{G_a})_{D^l}\right)\right)$$

where $\boldsymbol{r}_a:K_{l,a}\to\dot\bigcup_{a\in A}(\prod_{b_a\in B_a}B_{S_{r_{b_a}}})$ is defined by $\boldsymbol{r}_a(x)=(\boldsymbol{r}_{b_a}(x))_{b_a\in B_a}$ and $\bar{h}:E_{\eta_{l,a}}\to(E_{G_a})_{D^l}$ is the function such that the following diagram commutes:

$$\begin{tikzcd}
TK_{l,a}=E_{\eta_{l,a}} \arrow[r, "\bar{h}"] \arrow[d, "p_{\eta_{l,a}}"'] & (E_{G_a})_{D^l} \arrow[d, "(p_{G_a})_{D^l}"] \\
K_{l,a} \arrow[r, "h"'] & B_{G_a}
\end{tikzcd} \tag{3.3}$$

where h is the classifying function for the closed disk bundle $(E_{\eta_{l,a}}, K_{l,a}, p_{\eta_{l,a}}) = \eta_{l,a}$. Since $\partial TK_l = \partial(\overline{M - TK_l})$ and $f|_{f^{-1}(\partial(\overline{M-TK_l}))} = (\boldsymbol{c}_0|_{\partial(\overline{M-TK_l})})^*\gamma(l-1)$ (the pullback of $\gamma(l-1)$ under $\boldsymbol{c}_0|_{\partial(\overline{M-TK_l})}$), and $f|_{f^{-1}(\partial(\overline{TK_l}))} = ((\boldsymbol{c}_0 \circ \boldsymbol{c}')|_{\partial(\overline{TK_l})})^*\gamma(l-1)$ (the pullback of $\gamma(l-1)$ under the composition of $\boldsymbol{c}_0$ with $\boldsymbol{c}'$). It shows that $\boldsymbol{c}_0|_{(\partial(\overline{M-TK_l}) = \partial TK_l)}$ is homotopic with $(\boldsymbol{c}_0 \circ \boldsymbol{c}')|_{\partial TK_l}$. Without loss of generality, we could assume that these restriction functions are exactly the same function over ∂TK_l and then the function $\boldsymbol{c}'$ is a continuous well-defined function.

Furthermore, $\boldsymbol{c}'$ classifies f, i.e., f is the pullback in the following commutative diagram:

$$\begin{array}{ccc} \widetilde{M} & \longrightarrow & E(l) \\ {\scriptstyle \boldsymbol{c}^*\gamma(l)=f}\big\downarrow & & \big\downarrow{\scriptstyle \gamma(l)} \\ M & \xrightarrow[\boldsymbol{c}]{} & BR_k(l) \end{array} \tag{3.4}$$

Now let $c_1^*\gamma(l)$ and $c_2^*\gamma(l)$ be two pullbacks that are concordant. Using their concordance (that is also a branched covering of the same type l) it is not difficult to construct a homotopy between the maps $c_1 : M \to BR_k(l)$ and $c_2 : M \to BR_k(l)$. Hence, $(E(l), BR_k(l), \gamma(l))$ is the universal k-fold branched covering of type l. □

4. Homotopy groups

Recall that the definition of classifying spaces implies that the n-homotopy groups can be interpreted geometrically as concordance classes of branched coverings over the n-sphere. The construction of classifying spaces and the geometrical interpretation of the homotopy groups is exploited here to give information about some particular homotopy groups.

Let M, N be smooth manifolds, let $f : M \to N$ be a k-fold branched covering of type 2, i.e., its branch set K_2 is a submanifold of N of codimension 2. For each component of K_2 we can associate a partition $\sum_{i=1}^m k_i s_i = k$ of k. Let P be a set of partitions of k. Let us denote by $BR_k^P(2)$ the classifying space of k-fold branched coverings of type 2 for which the associated partitions belong to P.

In [2, p. 235]) Brand gave an explicit construction of the classifying space $BR_k^P(2)$. Inductively (using similar methods to those above), we extend this definition for any natural number l greater than 1.

In other words, we define:

$$BR_k^P(l) = \left(\bigcup_{a \in A} \left(\prod_{b_a \in B_a} B_{S_{r_{b_a}}} \times (E_{G_{b_a}})_{D^l} \right) \right) \cup_{\boldsymbol{c}} BR_k^P(l-1).$$

It is clear that for every natural number l greater than 1, $BR_k^P(l) \subset BR_k^P(l+1)$. So, for every $i \in \mathbb{N}$ and $l \geqslant 2$ we get a homomorphism of homotopy groups:

$$v_{i,l} : \pi_i(BR_k^P(l), *) \to \pi_i(BR_k^P(l+1), *),$$

where $*$ is a base point in $BR_k^P(2)$. Notice that if $f: M \to N$ is a k-fold branched covering of type $l+1$ classified by $g: N \to BR_k^P(l+1)$ then f is concordant to a k-fold branched covering of type l if and only if there is a map $g': N \to BR_k^P(l)$ so that the composition $N \xrightarrow{g'} BR_k^P(l) \hookrightarrow BR_k^P(l+1)$ is homotopic to g.

It is not difficult to prove that (see [9, p. 61]): For every l greater than i the homomorphism $v_{i,l}$ is bijective. Moreover, for l equal to i the homomorphism $v_{i,l}$ is surjective.

As a consequence, we have the following sequences of functions:

(a) $\Pi_1(BR_k^P(2),*) \xrightarrow{\sim} \Pi_1(BR_k^P(3),*) \xrightarrow{\sim} \Pi_1(BR_k^P(4),*) \xrightarrow{\sim} \cdots$

(b) $\Pi_2(BR_k^P(2),*) \twoheadrightarrow \Pi_2(BR_k^P(3),*) \xrightarrow{\sim} \Pi_2(BR_k^P(4),*) \xrightarrow{\sim} \cdots$

(c) $\Pi_3(BR_k^P(3),*) \twoheadrightarrow \Pi_3(BR_k^P(4),*) \xrightarrow{\sim} \Pi_3(BR_k^P(5),*) \xrightarrow{\sim} \cdots$

We notice that depending on the partition P the surjective homomorphism

$$v_{2,2}: \pi_2(BR_k^P(2)) \to \pi_2(BR_k^P(3)),$$

could be either bijective or not. For example, if $P = \{\{k\}\}$ then $v_{2,2}$ is bijective (see [9, p. 63]), in other words, if $P = \{\{k\}\}$ then for every l, $\Pi_2(BR_k^P(l)) \cong \Pi_2(BR_k^P(l+1))$. In this case, it is also true that for every i, $\Pi_i(BR_k^P(2)) \cong \Pi_i(BR_k^P(3))$. However, the next example shows that $\pi_2(BR_k^P(2))$ is not always isomorphic with $\pi_2(BR_k^P(3))$.

Let $f: M^2 \to N^2$ be a k-fold branched covering of type 2, with branch set a finite set $K_2 = \{x_1, \ldots, x_r\}$, and such that for each point $y \in \widetilde{K}_2 = f^{-1}(K_2)$ there is a neighbourhood U of y in M^2 on which the restricted map $f|_U : U \to f(U)$ is equivalent up to homeomorphism to the map $z \to z^n$ of the complex numbers for some integer $n \geqslant 1$. The number n is called the *ramification number*. We may associate to the k-fold branched covering f a branch data $D = \{A_1, A_2, \ldots, A_r\}$, where $A_i = [n_{i1}, n_{i2}, \ldots, n_{ij_i}]$ is a partition of k, and the numbers n_{ij} are the ramification numbers of the points in the fiber $f^{-1}(x_i)$.

Example 4.1. If $\{\{4\}, \{2,1,1\}\} \subset P$ then the homomorphism:

$$v_{2,2}: \pi_2(BR_k^P(2),*) \to \pi_2(BR_k^P(3),*)$$

is not injective.

In fact, using the methods developed by Gersten in [5], we can see that there exists a 4-fold branched covering $g: S^2 \to S^2$ of type 2 such that the branch data of g is $\{[4],[2,1,1],[2,1,1],[2,1,1]\}$ (actually, we have already made an explicit construction of it in the Examples 1.2(b) and (c)). This means that the branch set of g, $K_2 = \{x,y,z,w\}$ is such that $g^{-1}(x)$ has only one element with ramification number 4, and each of the sets $g^{-1}(y)$, $g^{-1}(z)$, and $g^{-1}(w)$ has only one element with ramification number 2 and two with ramification number 1. Let us extend the function $g: S^2 \to S^2$ radially to a function $f_0: D^3 \to D^3$ in the following way:

$$f_0(z) = |z|^3 g\left(\frac{z}{|z|}\right), \quad \forall z \in D^3.$$

So, f_0 is a 4-fold branched covering of type 3 from D^3 to D^3. Fig. 5 illustrates the function f_0, where the lines represent the branching set, the notation $x[4]$ means that

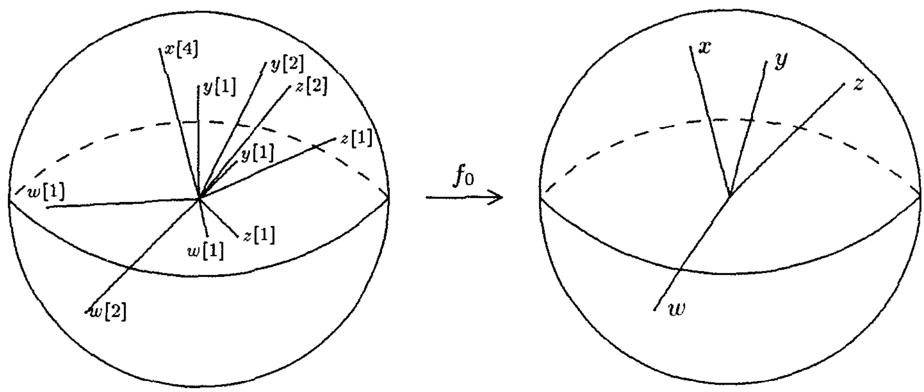

Fig. 5.

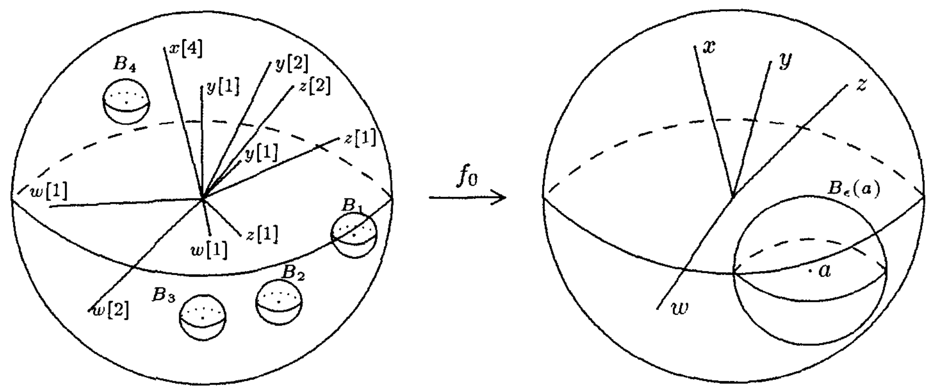

Fig. 6.

$f_0(x[4]) = x$, and the ramification number of $x[4]$ is 4. Now, let $a \in \mathrm{int}(D^3)$ such that a is not a branched point (see Fig. 6), and let us cut out a small open ball $B_\varepsilon(a)$ around the point a. Since a is not a branched point, we can choose the ball $B_\varepsilon(a)$ such that it does not intersect the branch set of f_0 . Therefore, $f_0^{-1}(B_\varepsilon(a))$ consists of four disjoint sets each one homeomorphic to $B_\varepsilon(a)$. Let us call these sets by B_i, $i = 1, \ldots, 4$, i.e., $f_0(B_1 \cup B_2 \cup B_3 \cup B_4) = B_\varepsilon(a)$.

Hence, it is clear that the function $f_0|_{(D^3 - f_0^{-1}((B_\varepsilon(a)))}$ is equivalent up to homeomorphism to a k-fold branched covering $F: W \to S^2 \times I$ of type 3, such that the restriction $F|_{F^{-1}(S^2 \times \{1\})}$ is equivalent up to homeomorphism to $g: S^2 \to S^2$, and the restriction $F|_{F^{-1}(S^2 \times \{0\})}$ is a 4-fold covering (it has empty branch set). Let us denote $F|_{F^{-1}(S^2 \times \{0\})}$ by h. So, g is concordant to h under a 4-fold branched covering of type 3.

Now, let us prove that g is not concordant to h by a 4-fold branched covering of type 2. In fact, using branched type 2, we see that the branched component that contains the

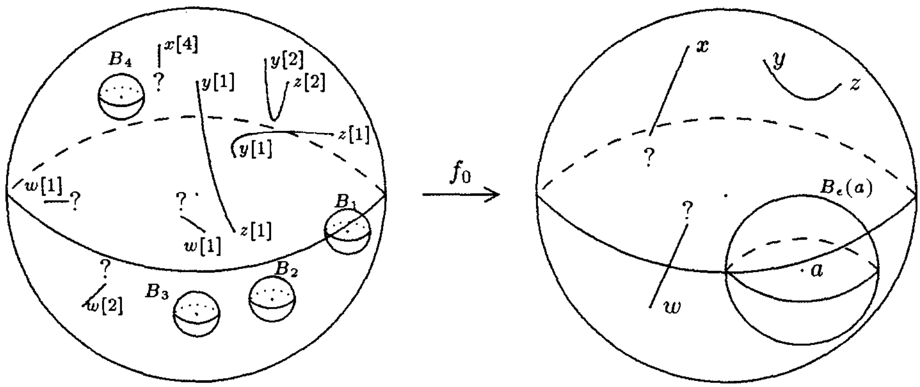

Fig. 7.

point $x[4]$ has to connect to another point with ramification number 4 belonging either to the domain of g or to the domain of h. Nevertheless, the only point with ramification number 4 inside these domains is $x[4]$. Moreover, if we assume that the points $y[2]$, $z[2]$ are in the same component, then the component that contains the point $w[2]$ cannot connect to any other point (see Fig. 7).

In particular the last example also says that for $P = \{\{4\}, \{2, 1, 1\}\}$ the second homotopy group $\pi_2(BR_k^P(2))$ is different from zero. Moreover, using the same techniques developed in the last example we can show that any k-fold branched covering of type n from S^n to S^n is concordant to a k-fold covering (with empty branch set) over S^n under a k-fold branched covering of type $n+1$, i.e., we have the following corollary:

Corollary. *All the k-fold branched coverings of type n from S^n to S^n are concordant of type $n+1$.*

Now let us consider only k-fold branched coverings of type l that verify the local condition specified in Section 2 and, moreover, their restrictions to the strata $\widetilde{K}_l$ are diffeomorphisms. Furthermore, we suppose that the groups $\widetilde{G}$ and G of coordinate transformations of the normal bundles of $\widetilde{K}_l$ and K_l are trivial groups. For this kind of branched coverings we get the following classifying space:

$$BR_k^P(l) = D^l \cup_c BR_k^P(l-1).$$

Applying classical techniques as Van Kampen's theorem, Mayer–Vietoris sequences and recalling that $\Pi_1(BR_k^P(2) \cong \mathcal{S}_k/\langle P\rangle$ (see [4]) we get the following examples:

Example 4.2. (i) If $k = 4$ and $P = \{\{2, 2\}\}$ then $\langle P\rangle = A_4$. So:

$$\Pi_1(BR_4^P(2)) \cong (\mathcal{S}_4/A_4) \cong \mathbb{Z}_2 \cong \widetilde{H}_1(BR_4^P(2)).$$

(ii) If $k = 5$ and $P = \{\{2, 2, 1\}\}$ then $\langle P\rangle = A_5$. Therefore:

$$\Pi_1(BR_5^P(2)) \cong (\mathcal{S}_5/A_5) \cong \mathbb{Z}_2 \cong \widetilde{H}_1(BR_5^P(2)).$$

(iii) If $k = 5$ and $P = \{\{4,1\},\{2,1,1,1\}\}$ then $\langle P\rangle = \mathcal{S}_5$. So:

$$\Pi_1\big(BR_5^P(2)\big) \cong (\mathcal{S}_5/\mathcal{S}_5) \cong \{0\} \cong \widetilde{H}_1\big(BR_5^P(2)\big).$$

(iv) If $k = 4$ and $P = \{\{3,1\}\}$ then $\langle P\rangle >= A_4$. Hence:

$$\Pi_1\big(BR_4^P(2)\big) \cong (\mathcal{S}_4/A_4) \cong \mathbb{Z}_2 \cong \widetilde{H}_1\big(BR_4^P(2)\big).$$

(v) If $k = 4$ and $P = \{\{4\},\{2,2\},\{3,1\}\}$ then $\langle P\rangle = \mathcal{S}_4$. So:

$$\Pi_1\big(BR_4^P(2)\big) \cong (\mathcal{S}_4/\mathcal{S}_4) \cong \{0\} \cong \widetilde{H}_1\big(BR_4^P(2)\big).$$

As in the example studied at the beginning of the section, we get for Examples 4.2(iv) and 4.2(v) that:

$$\Pi_2\big(BR_4^P(2)\big) \neq 0.$$

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References

[1] J.W. Alexander, Note on Riemann spaces, Bull. Amer. Math. Soc. 26 (1920) 370–372.
[2] N. Brand, Classifying spaces for branched coverings, Indiana Univ. Math. J. 29 (1980) 229–248.
[3] N. Brand, Homotopy theory of branched coverings, Thesis, Stanford University (1978).
[4] N. Brand and G. Brumfiel, Concordance classes of simple branched covers, Preprint.
[5] S.M. Gersten, On branched covers of the 2-sphere by the 2-sphere, Proc. Amer. Math. Soc. 101 (1987) 761–766.
[6] V. Guillemin and A. Pollack, Differential Topology (Prentice-Hall, Englewood Cliffs, NJ, 1974).
[7] D. Husemoller, Fibre Bundles (Springer, New York, 1975).
[8] N.E. Steenrod, Topology of Fibre Bundles, Princeton Math. Ser. (Princeton University Press, Princeton, NJ, 1975).
[9] D.M. Tejada, Universal branched coverings, Thesis, University of North Texas (1993).
[10] D.M. Tejada, Brown's representability theorem and branched coverings, Rev. Colombiana Mat. 28 (1994) 23–34.